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LATTICE-UNIVERSAL ORLICZ SPACES ON PROBABILITY SPACES

BY

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ABSTRACT

Lattice-universal Orlicz function spaces $L^{F_{\alpha,\beta}}[0,1]$ with prefixed Boyd indices are constructed. Namely, given $0 < \alpha < \beta < \infty$ arbitrary there exists Orlicz function spaces $L^{F_{\alpha,\beta}}[0,1]$ with indices α and β such that every Orlicz function space $L^G[0,1]$ with indices between α and β is lattice-isomorphic to a sublattice of $L^{F_{\alpha,\beta}}[0,1]$. The existence of classes of universal Orlicz spaces $\ell^{F_{\alpha,\beta}}(I)$ with uncountable symmetric basis and prefixed indices α and β is also proved in the uncountable discrete case.

1. Introduction

The existence or non-existence of universal spaces inside predetermined classes of Banach spaces has been studied intensely in many contexts. For example, we mention that there does not exist a separable super-reflexive Banach space universal for the class of all ℓ^p-spaces for $1 < p < \infty$, and that there is no reflexive Banach space universal for the class of all separable reflexive Banach spaces ([Bo]). On the positive result side, every separable Banach lattice and every rearrangement invariant Banach space is isometrically lattice-isomorphic to a sublattice of the Banach envelope of Weak-L^1 ([Lo-P], [Le]).

In this paper we are interested in finding universal classical separable function spaces with prefixed upper and lower estimates. In the setting of *Orlicz sequence*

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spaces the existence of universal sequence spaces was proved by Lindenstrauss and Tzafriri ([L-T1]) by finding Orlicz sequence spaces $\ell^{F_{\alpha,\beta}}$, with arbitrary prefixed indices $1 \leq \alpha < \beta < \infty$, in which every Orlicz sequence space ℓ^G with indices between α and β is isomorphic to a (complemented) subspace of the space $\ell^{F_{\alpha,\beta}}$. Universal *Orlicz function spaces* on the *unbounded* interval $(0,\infty)$ were studied in ([H-Ru]) showing that the spaces $L^\alpha + L^\beta$ are lattice-universal for the class of all Orlicz function spaces $L^G(0,\infty)$ with Boyd indices strictly between α and β, i.e. every space $L^G(0,\infty)$ is lattice-isomorphic to a sublattice of the space $L^\alpha + L^\beta$. The embedding behavior is varied in the extreme cases of spaces with one index equal to α or to β. The methods of proofs make use of some results given by Bretagnolle and Dacunha-Castelle ([B-D]), and in the Memoirs of Johnson, Maurey, Schechtman and Tzafriri ([J-M-S-T]) and Kalton ([K2]) concerning the symmetric structure of rearrangement invariant spaces.

In the context of function spaces on the *probability* space $[0,1]$, the existence or non-existence of universal Orlicz function spaces $L^F[0,1]$ with prefixed indices has remained open: are there Orlicz spaces $L^F[0,1]$, with indices α and β, in which every other Orlicz space $L^G[0,1]$ with indices between α and β can be isomorphically embedded or even lattice-isomorphically embedded? (cf. [H-Ru] page 98). We point out that while isomorphic embeddings of scales of L^p-spaces for $p \leq 2$ into rearrangement invariant spaces on $[0,1]$ are well-known and appear very often (using f.i. Poisson process, cf. [L-T3] Theorem 2.f.4), there is, however, a shortage in the opposite case $p > 2$. Indeed first examples of separable rearrangement invariant spaces on $[0,1]$ containing subspaces isomorphic to L^p-spaces for different scalars $p > 2$ have been given in ([H-R2]). This rigid and limited behavior in the case $p > 2$ is due to the following remarkable fact: the existence of an isomorphic embedding of an L^p-space for $p > 2$ into a separable rearrangement invariant space E on $[0,1]$ implies that there exists also a *lattice*-isomorphic embedding of L^p into E ([H-K] Corollary 7.4).

One of the purposes of this paper is to answer in the positive the above question, showing the existence of classes of Orlicz function spaces $L^{F_{\alpha,\beta}}[0,1]$ with prefixed indices $0 < \alpha < \beta < \infty$, which are lattice-universal for all Orlicz function spaces with indices between α and β. Thus the following will be proved:

THEOREM 1: *Let $0 < \alpha < \beta < \infty$. There exists an Orlicz function space $L^{F_{\alpha,\beta}}[0,1]$, with indices $\alpha^\infty_{F_{\alpha,\beta}} = \alpha$ and $\beta^\infty_{F_{\alpha,\beta}} = \beta$, such that every α-convex β-concave Orlicz function space $L^G[0,1]$ is lattice-isomorphic to a sublattice of $L^{F_{\alpha,\beta}}[0,1]$.*

In particular, the above spaces $L^{F_{\alpha,\beta}}[0,1]$ contain a lattice-isomorphic copy of

$L^p[0,1]$ for every $\alpha \leq p \leq \beta$. The behavior in the boundary indices cases is also analyzed.

Next, we construct scales of universal Orlicz function spaces $L^F[0,1]$ depending also on a third parameter γ, between α and β, which gives the **upper inclusion index** γ_F^∞ of the function F at ∞,

$$\gamma_F^\infty = \limsup_{x \to \infty} \frac{\log F(x)}{\log x} = \inf\{p > 0 : L^p[0,1] \subset L^F[0,1]\}.$$

This index gives a lower bound for the lattice-isomorphic embedding, since $\gamma_F^\infty \leq p \leq \beta_F^\infty$ occurs whenever $L^p[0,1]$ is lattice-isomorphic to a sublattice of $L^F[0,1]$. Thus the universal spaces $L^{F_{\alpha,\beta}}[0,1]$ of Theorem 1 above satisfy $\gamma_{F_{\alpha,\beta}}^\infty = \alpha$. It is known that the set of scalars p's for which an Orlicz space $L^F[0,1]$ contains an isomorphic copy of $L^p[0,1]$ is not necessarily a closed set ([H-R2]). Given two Orlicz functions F and G, by $F \prec G$ at ∞ (resp. at 0) we mean that there exists a positive constant C such that $F(x) \leq CG(x)$ for $x \geq 1$ (resp. $0 \leq x \leq 1$).

THEOREM 2: *Let $0 < \alpha < \gamma < \beta < \infty$. There exists an α-convex Orlicz function space $L^{F_{\alpha,\beta,\gamma}}[0,1] \equiv L^F[0,1]$ with indices $\alpha_F^\infty = \alpha$, $\beta_F^\infty = \beta$ and $\gamma_F^\infty = \gamma$ such that $L^F[0,1]$ contains a sublattice lattice-isomorphic to $L^G[0,1]$ for every α'-convex β-concave Orlicz function G at ∞ with $\alpha < \alpha' \leq \beta$ and $x^\gamma \prec G$ at ∞.*

In particular, every γ-convex β-concave Orlicz function space $L^G[0,1]$ is lattice-isomorphically embedded into the above space $L^{F_{\alpha,\beta,\gamma}}[0,1]$. In order to get this function space result, we solve first similar questions in the context of uncountable discrete spaces. Later on we use some transference arguments and the criteria for the lattice isomorphic embedding of Orlicz function spaces into a space $L^F[0,1]$ given in ([J-M-S-T] Theorem 7.7) in terms of the associated set $\sum_{F,1}^\infty$ of all the Orlicz functions equivalent at ∞ to a function

$$H(x) = \int_0^\infty \frac{F(xs)}{F(s)} d\mu(s) \quad (x \geq 1)$$

where μ is a probability measure on $(0,\infty)$ satisfying $\int_0^\infty \frac{1}{F(s)} d\mu(s) \leq 1$.

Thus we also study the existence of universal Orlicz spaces $\ell^F(I)$ with an uncountable symmetric basis and prefixed indices. The structure of Banach spaces with an uncountable symmetric basis has been studied by Troyanski ([T1], [T2]) and Drewnowski ([D]), and the quasi-Banach case in ([H-T]). It has been proved in ([H-R2]) that there exist Orlicz spaces $\ell^F(I)$ with an uncountable symmetric basis containing isomorphically scales of different $\ell^p(\Gamma)$-spaces for uncountable sets $\Gamma \subset I$. Here we prove the following general result:

THEOREM 3: *Let $0 < \alpha < \beta < \infty$. There exists an Orlicz space $\ell^{F_{\alpha,\beta}}(I)$, with indices $\alpha_{F_{\alpha,\beta}} = \alpha$ and $\beta_{F_{\alpha,\beta}} = \beta$, such that $\ell^{F_{\alpha,\beta}}(I)$ contains an isomorphic copy of any α-convex β-concave Orlicz space $\ell^G(\Gamma)$ with $\Gamma \subset I$.*

In particular, these convex functions $F_{\alpha,\beta}$ (for $\alpha > 1$) provide examples of universal Orlicz *sequence* spaces different from those given by Lindenstrauss and Tzafriri in [L-T1] (see below Remark 3.3). We also construct scales of universal spaces $\ell^F(I)$ with a prefixed **lower inclusion index** γ_F, between α and β, where

$$\gamma_F = \liminf_{x\to 0} \frac{\log F(x)}{\log x} = \sup\{p > 0 : \ell^p(I) \subset \ell^F(I)\}.$$

If $\ell^p(\Gamma)$ is isomorphically embedded into $\ell^F(I)$, where $\Gamma \subset I$ are uncountable sets, then $\alpha_F \le p \le \gamma_F$. Hence, the universal spaces $\ell^{F_{\alpha,\beta}}(I)$ above satisfy $\gamma_{F_{\alpha,\beta}} = \beta$. In general, the set of scalar p's for which an Orlicz space $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$, $\Gamma \subset I$ uncountable sets, is not a closed set ([H-R2]).

THEOREM 4: *Let $0 < \alpha < \gamma < \beta < \infty$. There exists a β-concave Orlicz space $\ell^{F_{\alpha,\beta,\gamma}}(I) \equiv \ell^F(I)$ with indices $\alpha_F = \alpha, \beta_F = \beta$ and $\gamma_F = \gamma$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^G(\Gamma)$ for sets $\Gamma \subset I$ and every α-convex β'-concave Orlicz function G at 0 with $\alpha \le \beta' < \beta$ and $x^\gamma \prec G$ at 0.*

In particular, every α-convex γ-concave Orlicz space $\ell^G(\Gamma)$ is isomorphically embedded into the space $\ell^{F_{\alpha,\beta,\gamma}}(I)$ above. The construction of these universal Orlicz functions at 0 is carried out with the help of a combinatorial technique developed in previous works ([H-T], [H-R1], [H-R2]). We also use the criteria for isomorphic embeddings of spaces $\ell^G(\Gamma)$ into an Orlicz space $\ell^F(I)$ ([R] Theorem B, [H-T] Proposition 5): An Orlicz space $\ell^F(I)$ contains an isomorphic copy of $\ell^G(\Gamma)$ for uncountable sets $\Gamma \subset I$, if and only if $G \in \sum_{F,1}$ where $\sum_{F,1}$ is the set of all Orlicz functions equivalent at 0 to a function

$$H(x) = \int_0^1 \frac{F(xs)}{F(s)} d\mu(s) \quad (0 \le x \le 1)$$

where μ is a probability measure on $[0,1]$.

The paper is organized as follows. In Section 2 we give some preliminary technical Lemmas. Section 3 contains the proofs of the main results in the discrete case. Finally, in Section 4 we construct the universal function spaces on $[0,1]$. The notation is standard and we refer to the monographs ([L-T2], [L-T3], [K-R]) for unexplained definitions.

2. Preliminary results

In this section we give some required technical results which will be used in the proofs of Theorems 3 and 4. We start by recalling two lemmas stated in ([H-R2], pp. 195–6), which are basic in these constructions.

LEMMA 5: *Given $(h_i)_{i=0}^{\infty}$ a sequence of positive integers with $h_0 = 1$, there exist two integer sequences $(k_i)_{i=0}^{\infty}$ and $(m_i)_{i=0}^{\infty}$ with $m_i > k_i = \sum_{j=0}^{i-1} m_j$ for $j = 1, 2, \dots$ and $\lim_{i\to\infty}(m_{i+1} - m_i) = \infty$, such that*

$$\sum_{i=0}^{\infty} f(n + k_i) = h_n \quad (n = 0, 1, 2, \dots)$$

and

$$\sum_{i=0}^{\infty} f(k_i - n) \le (n + 2)^2 \quad (n = 0, 1, 2, \dots)$$

where f is the function $f = \sum_{i=0}^{\infty} \chi_{[m_i, m_i+1)}$.

Notice that in other form the above lemma states that, given a sequence of positive integers $(h_i)_{i=0}^{\infty}$ with $h_0 = 1$, there exists a set of couples of positive integers $\{(m_j, k_i)\}$ with $m_i > k_i$ such that for each positive integer n: (i) there exist precisely h_n couples (m_j, k_i) such that $m_j - k_i = n$; (ii) there exist at most $(n+2)^2$ couples (m_j, k_i) such that $k_i - m_j = n$.

Given $\delta > 0$, we consider the sequences (k_i) and (m_i) constructed in the above Lemma 5 for the case of the sequence (h_n) equal to $([2^{n\delta}])$([ ] denotes the integral part); and we define the sequences (α_n) and (δ_n) by $\alpha_{k_i} = 2^{k_i\delta}$; $\delta_{m_i} = 1$ for $i = 0, 1, \dots$ and $\alpha_j = \delta_{j'} = 0$ in the remaining cases. Then, using Lemma 5, we have

LEMMA 6: *Given $\delta \ge 0$, there exist two positive sequences $(\alpha_n)_{n=0}^{\infty}$ and $(\delta_n)_{n=0}^{\infty}$ such that*

$$\sum_{n=0}^{\infty} \alpha_n 2^{-\delta n} = \infty, \quad \frac{1}{2} \le \sum_{n=0}^{\infty} \alpha_n \delta_{n+k} 2^{-(n+k)\delta} \le 1$$

and

$$\sum_{n=k}^{\infty} \alpha_n \delta_{n-k} 2^{-(n-k)\delta} \le (k+2)^2 2^{\delta k}$$

for every $k \in \mathbb{N}$.

The following result extends Lemma 6 in [H-R2]

LEMMA 7: *Let g be a positive function on $[0,1]$ with $g(0) = 0$ and right-continuous at 0. Given $\delta > 0$, there exists a sequence of positive numbers $(\alpha'_n)_{n=0}^{\infty}$ and an integer $k_0 > 0$ such that*

$$g(2^{-k}) \leq \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \leq g(2^{-k}) + \frac{2}{2^{k\delta}}$$

for every $k \geq k_0$.

Proof: We apply Lemma 5 for the sequence $(h_n) = ([2^{n\delta}])$, finding the associated sequences $(k_i)_{i=0}^{\infty}$ and $(m_i)_{i=0}^{\infty}$. Thus, for every natural n there exist exactly $[2^{n\delta}]$ couples (k_i, m_j) such that $m_j - k_i = n$. Using the right continuity of the function g at 0 we find an integer $k_0 > 0$ such that $[2^{k\delta}g(2^{-k})]+1 < [2^{k\delta}]$ for every $k \geq k_0$. Let us denote by A_n the set of the $[2^{n\delta}]$ natural numbers k_i that appear in these pairs (k_i, m_j) with $m_j - k_i = n$, and by A'_n we denote the subset of A_n given by the last $[2^{k\delta}g(2^{-k})] + 1$ natural numbers k_i which are in A_n.

Let us show that the set

$$B_k = \bigcup_{\substack{n=k_0 \\ n\neq k}}^{\infty} A'_n \cap A_k$$

contains at most one element. Indeed, if $k_i \in B_k$ there exists m_j such that $m_j - k_i = k$, and hence $j \geq i$. Now, if $j > i$ it follows from the construction in Lemma 5 that $m_{j-1} < k < m_j$, hence there is at most a natural j verifying it. Let us suppose now that $j = i$, which means $k = m_i - k_i = l_i$ and $k_i \in A'_n$ for some $n \geq k_0$ with $n \neq k$. Then there exists $j' > i$ such that $n = m_{j'} - k_i$. Hence $n > k$. Let us show now that k_i is the first element of A_n, which contradicts that $k_i \in A'_n$. If $n = m_{i'} - k_{i'} = l_{i'}$ as $n > k$, we have $i' > i$ and hence $k_{i'} > k_i$. Thus k_i is the first element of A_n.

We consider now the sequence $(\alpha'_n)_{n=0}^{\infty}$ defined by $\alpha'_n = 2^{n\delta}$ if $n \in \bigcup_{h=0}^{\infty} A'_h$, and $\alpha'_n = 0$ otherwise. Since $A_k \cap \bigcup_{h=k_0}^{\infty} A'_h = A'_k \cup B_k$ we have

$$\sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} = \sum_{n \in A'_k \cup B_k} 2^{n\delta} \delta_{n+k} 2^{-(n+k)\delta}$$

for $k \geq k_0$. Hence

$$\sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \leq \frac{[2^{k\delta}g(2^{-k})] + 2}{2^{k\delta}} \leq g(2^{-k}) + \frac{2}{2^{k\delta}}$$

and

$$\sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \geq \frac{[2^{k\delta} g(2^{-k})] + 1}{2^{k\delta}} \geq g(2^{-k})$$

for every $k \geq k_0$. ∎

Remark: The above defined sequence $(\alpha'_n)_{n=1}^{\infty}$ satisfies $0 \leq \alpha'_n \leq \alpha_n$ where (α_n) is the sequence considered in Lemma 6. Then

$$(+) \qquad \sum_{n=k}^{\infty} \alpha'_n \delta_{n-k} 2^{-(n-k)\delta} \leq (k+2)^2 2^{\delta k}$$

also holds for every $k \in \mathbb{N}$.

Given $\varepsilon > 0$ we consider the associated sequences (δ_n) and (m_i) defined in Lemmas 5 and 6 for $\delta = \varepsilon$. Let the sequence $(c_k) = ((k+1)^4)_{k=1}^{\infty}$ and $M = \{m_i\}_{i=0}^{\infty}$. For each $k \in \mathbb{N}$ we denote

$$M_k = (M+k) \setminus \bigcup_{j=0}^{k-1} (M+j).$$

Let us define the sequence (ε_n) by $\varepsilon_0 = 0$, and

$$\varepsilon_n = \begin{cases} 2^{-n\varepsilon} & \text{if } n \in M_0 = M, \\ 2^{-n\varepsilon} c_k^{-1} = 2^{-n\varepsilon} c_k^{-1} \delta_{n-k} & \text{if } n \in M_k. \end{cases}$$

Lemma 8: *Let $\varepsilon > 0$, $(c_n) = ((n+1)^4)$, and (ε_n) be the sequence defined above. There exist positive constants A and B and a sequence $(\alpha_n)_{n=0}^{\infty}$ of positive numbers such that*

$$A \leq \sum_{n=0}^{\infty} \alpha_n \varepsilon_{n+k} \leq B$$

for every $k \in \mathbb{N}$.

Proof: See ([H-R2] p. 198). ∎

Lemma 9: *Let $\varepsilon > 0$, $(c_n) = ((n+1)^4)$, and (ε_n) be the sequence defined above. If g is a positive function on $[0,1]$ with $g(0) = 0$ and right-continuous at 0, then there exist positive constants A and B and a sequence of positive numbers (α'_n) such that*

$$Ag(2^{-k}) \leq \sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} \leq B\left(\sum_{i=0}^{k} \frac{g(2^{-k+i})}{2^{i\varepsilon} c_i} + \frac{1}{2^{k\varepsilon}}\right)$$

for every natural $k = 0, 1, \ldots$.

Proof: We proceed in an analogous way to Lemma 7 in [H-R2]. Using above Lemma 7 for $\delta = \varepsilon$, there exists an integer $k_0 > 0$ and a sequence of positive number (α'_n) such that

$$\sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} \geq \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\varepsilon} \geq g(2^{-k})$$

for every $k \geq k_0$. On the other hand, $\sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} = \sum_{i=0}^{\infty} \sum_{n+k \in M_i} \alpha'_n \varepsilon_{n+k}$. Now by using Lemma 7 we have, for $k \geq k_0$,

$$\begin{aligned}
\sum_{i=0}^{k} \sum_{n+k \in M_i} \alpha'_n \varepsilon_{n+k} &\leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k-i} 2^{-(n+k)\varepsilon} \frac{1}{c_i} \\
&\leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k-i} \frac{2^{-(n+k-i)\varepsilon}}{2^{i\varepsilon} c_i} \\
&\leq \sum_{i=0}^{k} \frac{1}{2^{i\varepsilon} c_i} \left(g(2^{-(k-i)}) + \frac{2}{2^{(k-i)\varepsilon}} \right) \\
&\leq \sum_{i=0}^{k} g(2^{-k+i}) \frac{1}{2^{i\varepsilon} c_i} + \frac{C'}{2^{k\varepsilon}}
\end{aligned}$$

where $C' = \sum_{i=0}^{\infty} \frac{2}{c_i} < \infty$; and

$$\begin{aligned}
\sum_{i=k+1}^{\infty} \sum_{n+k \in M_i} \alpha'_n \varepsilon_{n+k} &\leq \sum_{i=k+1}^{\infty} \sum_{n} \alpha'_n \frac{\delta_{n+k-i}}{2^{i\varepsilon} c_i} 2^{-(n+k-i)\varepsilon} \\
&\leq \sum_{i=k+1}^{\infty} \frac{(i-k+2)^2}{2^{i\varepsilon} c_i} 2^{(i-k)\varepsilon} \\
&\leq \sum_{i=k+1}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{(i-k+2)^2}{c_i} \leq \frac{1}{2^{k\varepsilon}} \sum_{i=1}^{\infty} \frac{(i+2)^2}{c_i} \leq \frac{C''}{2^{k\varepsilon}}
\end{aligned}$$

which concludes the proof, since we can replace the constants in order to get the inequality for every $k \geq 0$. ∎

Given $\varepsilon > \delta \geq 0$, we consider the sequence (ε_n) defined by $\varepsilon_0 = 0$, and

$$\varepsilon_n = \begin{cases} 2^{-n\delta} & \text{if } n \in M = M_0 \\ c^{-k} 2^{-(n-k)\delta} = c^{-k} 2^{-(n-k)\delta} \delta_{n-k} & \text{if } n \in M_k \end{cases}$$

where $c = 2^{\varepsilon} > 1$.

LEMMA 10: *Let $\varepsilon > \delta \geq 0$ and (ε_n) be the sequence defined above. If g is a positive function on $[0,1]$ with $g(0) = 0$ and right-continuous at 0, then there exist positive constants A_1 and B_1, and a sequence of positive numbers (α'_n) such that*

$$A_1 g(2^{-k}) \leq \sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} \leq B_1\Big(\sum_{i=0}^{k} \frac{g(2^{-k+i})}{2^{i\varepsilon}} + \frac{1}{2^{k\delta}}\Big)$$

for every natural $k = 0, 1, 2, \ldots$.

Proof: Using Lemma 7 we have, for $k \geq k_0$,

$$\sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} \geq \sum_{n+k\in M} \alpha'_n \varepsilon_{n+k} = \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \geq g\Big(\frac{1}{2^k}\Big).$$

On the other hand, $\sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} = \sum_{i=0}^{\infty} \sum_{n+k\in M_i} \alpha'_n \varepsilon_{n+k}$. Now

$$\begin{aligned}
\sum_{i=0}^{k} \sum_{n+k\in M_i} \alpha'_n \varepsilon_{n+k} &\leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \frac{\alpha'_n \delta_{n+k-i} 2^{-(n+k-i)\delta}}{2^{i\varepsilon}} \\
&\leq \sum_{i=0}^{k} \frac{1}{2^{i\varepsilon}} (g(2^{-k+i}) + \frac{2}{2^{(k-i)\delta}}) \\
&\leq \sum_{i=0}^{k} \frac{1}{2^{i\varepsilon}} g(2^{-k+i)} + \frac{2}{2^{k\delta}} \sum_{i=0}^{\infty} \frac{1}{2^{i(\varepsilon-\delta)}} \\
&= \sum_{i=0}^{k} \frac{g(2^{-k+i})}{2^{i\varepsilon}} + \frac{C'}{2^{k\delta}}
\end{aligned}$$

where C' is a positive constant; and, using (+), we have

$$\begin{aligned}
\sum_{i=k+1}^{\infty} \sum_{n+k\in M_i} \alpha'_n \varepsilon_{n+k} &\leq \sum_{i=k+1}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha'_n \delta_{n+k-i} 2^{-(n+k-i)\delta}}{2^{i\varepsilon}} \\
&\leq \sum_{i=k+1}^{\infty} \frac{1}{2^{i\varepsilon}} (i-k+2)^2 2^{(i-k)\delta} \\
&= \sum_{j=1}^{\infty} \frac{(j+2)^2}{2^{(k+j)\varepsilon}} 2^{j\delta} \\
&= \frac{1}{2^{k\varepsilon}} \sum_{j=1}^{\infty} \frac{(j+2)^2}{2^{j(\varepsilon-\delta)}} \leq \frac{C''}{2^{k\delta}}.
\end{aligned}$$

Hence, there exist positive constants A_1 and B_1 such that

$$A_1 g\Big(\frac{1}{2^k}\Big) \leq \sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} \leq B_1\Big(\sum_{i=0}^{k} \frac{g(2^{-k+i})}{2^{i\varepsilon}} + \frac{1}{2^{k\delta}}\Big)$$

for every natural k. ∎

3. Discrete spaces

We are now in a position to prove the existence of spaces $\ell^F(I)$ with prefixed indices, which are universal for all Orlicz spaces $\ell^G(I)$ with indices between α and β.

THEOREM 11: *Let $0 < \alpha < \beta < \infty$. There exists an Orlicz space $\ell^{F_{\alpha,\beta}}(I) \equiv \ell^F(I)$, with indices $\alpha_F = \alpha$ and $\beta_F = \beta$, such that $\ell^F(I)$ contains an isomorphic copy of any α-convex β-concave Orlicz space $\ell^G(\Gamma)$ with $\Gamma \subset I$.*

Proof: We can reduce to consider the case $\alpha = 1 < \beta = 1 + \varepsilon$ for $\varepsilon > 0$ using the convexification properties of the sets $\sum_{F,1}$ (cf. [H-R1] p. 170). Let (ε_n) be the sequence considered in Lemmas 8 and 9. If f is the function $f = \sum_{n=0}^{\infty} \varepsilon_n \chi_{(2^{-(n+1)},2^{-n}]}$, we define on $[0,1]$ the function

$$F(x) = \int_0^x f(t)dt = \int_0^x \sum_{n=0}^{\infty} \varepsilon_n \chi_{(2^{-(n+1)},2^{-n}]}.$$

First we consider the case of G a strict 1-convex Orlicz function at 0, i.e. when G has the representation $G(x) = \int_0^x g(t)dt$, for g the right derivative of G, and g is a non-decreasing right-continuous function with $g(0) = 0$ and $g(t) > 0$ for $t > 0$ (cf. [Li]). Then, by Lemma 9, there exists a sequence (α'_n) of positive numbers and two positive constants A and B such that for $1/2^{k+1} < x \le 1/2^k$ we have

$$Ag(x) \le Ag\Big(\frac{1}{2^k}\Big) \le \sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} = \sum_{n=0}^{\infty} \alpha'_n f\Big(\frac{x}{2^n}\Big) \le B\Big(\sum_{i=0}^{k} \frac{g(2^{-k+i})}{2^{i\varepsilon}c_i} + \frac{1}{2^{k\varepsilon}}\Big),$$

and

$$\sum_{i=0}^{k} \frac{g(2^{-k+i})}{2^{i\varepsilon}c_i} \le \sum_{i=0}^{\infty} \frac{g(2^{-k+i})}{2^{i\varepsilon}c_i} \le \sum_{i=0}^{\infty} \frac{g(2^{i+1}x)}{2^{i\varepsilon}c_i},$$

hence

$$Ag(x) \le \sum_{n=0}^{\infty} \alpha'_n f\Big(\frac{x}{2^n}\Big) \le 2^{\varepsilon} B\Big(\sum_{i=0}^{\infty} \frac{g(2^{i+1}x)}{2^{\varepsilon(i+1)}c_i} + x^{\varepsilon}\Big)$$

for $0 \le x \le 1$. Then, by integration and the Beppo–Levi Theorem we have

$$AG(x) \le \sum_{n=0}^{\infty} \alpha'_n 2^n F\Big(\frac{x}{2^n}\Big) \le 2^{\varepsilon} B\Big(\sum_{i=0}^{\infty} \frac{G(2^{i+1}x)}{2^{(i+1)(1+\varepsilon)}} + \frac{x^{1+\varepsilon}}{(1+\varepsilon)}\Big)$$

for $0 \leq x < 1$. Now, by the $(1+\varepsilon)$-concavity of the function G at 0, there exists a positive constant C such that $G(rx) \leq Cr^{(1+\varepsilon)}G(x)$ for $r > 1$ and $0 < x < 1$, which implies that

$$\sum_{i=0}^{\infty} \frac{G(2^{i+1}x)}{2^{(i+1)(1+\varepsilon)}c_i} \leq G(x)C\Big(\sum_{i=1}^{\infty} \frac{1}{c_i}\Big).$$

Hence we deduce that there exist positive constants A_1 and B_1 such that for x near 0 we have

$$A_1 G(x) \leq \sum_{n=0}^{\infty} \alpha'_n 2^n F\Big(\frac{x}{2^n}\Big) \leq B_1 G(x).$$

Thus, if μ is the discrete probability measure on $[0,1]$ defined by $\mu(2^{-n}) = \alpha'_n 2^n F(2^{-n})$ we have that the function G is equivalent at 0 to the function

$$\widetilde{G}(x) = \int_0^1 \frac{F(xt)}{F(t)} d\mu(t),$$

hence $G \in \sum_{F,1}$ and, using ([R] Theorem B or [H-T] Proposition 5), we conclude that $\ell^F(I)$ contains a subspace isomorphic to $\ell^G(\Gamma)$ for every set $\Gamma \subset I$.

Let us consider now the case of the convex function G non-strict convex, i.e. when $g(0) > 0$. Then $G(x)$ is equivalent to the function x at 0. Now, by Lemma 8, there exists a sequence of positive numbers $(\alpha_n)_{n=0}^{\infty}$ and two positive constants A and B such that

$$A \leq \sum_{n=0}^{\infty} \alpha_n \varepsilon_{n+k} \leq B$$

for every $k \in \mathbb{N}$. Hence for $0 < x \leq 1$ we have

$$A \leq \sum_{n=0}^{\infty} \alpha_n f\Big(\frac{x}{2^n}\Big) \leq B$$

and, by integration and the Beppo–Levi Theorem, we deduce

$$Ax \leq \sum_{n=0}^{\infty} \alpha_n 2^n F\Big(\frac{x}{2^n}\Big) \leq Bx$$

for $0 \leq x \leq 1$. It follows that $x \in \sum_{F,1}$, hence $\ell^F(I)$ contains also an isomorphic copy of $\ell^1(\Gamma)$ for $\Gamma \subset I$.

Let us show now that the indices of F are $\alpha_F = 1$ and $\beta_F = 1+\varepsilon$. To prove $\alpha_F = 1$ let us first check that

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty$$

for every $q > 1$. Since

$$2^{-m}F(2^{-m}) = \sum_{i=0}^{\infty} \frac{\varepsilon_{m+i}}{2^{i+1}},$$

for $m = m_i - n > m_{i-1}$ we have $2^{m+n}F(2^{-m-n}) \geq \varepsilon_{m+n}/2$, and $2^m F(2^{-m}) \leq \varepsilon_m + \varepsilon_{m_i}$ with

$$\frac{\varepsilon_m}{\varepsilon_{m_i}} = \frac{2^{-m\varepsilon}}{2^{-m_i\varepsilon}} c_k^{-1}\delta_{n-k} = 2^{n\varepsilon}c_k^{-1}\delta_{n-k} \to 0$$

for $i \to \infty$ and n fixed. Then

$$\sup_m \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} \geq \frac{1}{2}2^{n(q-1)} \quad \text{and} \quad \sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty.$$

On the other hand, for $q < 1$ we have

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} < \infty$$

since

$$\sup_m \frac{2^{qn}F(2^{-n-m})}{F(2^{-m})} = \sup_m 2^{(q-1)n} \frac{\sum_{i=0}^{\infty} \varepsilon_{m+n+i} \frac{1}{2^{i+1}}}{\sum_{i=0}^{\infty} \varepsilon_{m+i} \frac{1}{2^{i+1}}} \leq C2^{(q-1)n} < \infty$$

for some constant $C > 0$.

Finally, let us show that $\beta_F = 1 + \varepsilon$. By using $\varepsilon_m \leq 2^{n\varepsilon}c_n\varepsilon_{m+n}$ for $m, n \in \mathbb{N}$, we have

$$\frac{2^{-n}F(2^{-m})}{F(2^{-m-n})} = \frac{\sum_{i=0}^{\infty} \frac{\varepsilon_{m+i}}{2^{i+1}}}{\sum_{i=0}^{\infty} \frac{\varepsilon_{m+n+i}}{2^{i+1}}} \leq c_n 2^{n\varepsilon},$$

which implies that $\beta_F \leq (1+\varepsilon)$. Conversely, we consider $m = m_i < m_{i+1} - n$; then $2^m F(2^{-m}) \geq \varepsilon_m/2$, and

$$2^{(m+n)}F(2^{-m-n}) \leq (\varepsilon_{m+n} + \varepsilon_{m_{i+1}}) \leq (2^{-\varepsilon n}\varepsilon_{m_i} + \varepsilon_{m_{i+1}}).$$

Hence, since

$$\frac{\varepsilon_{m_{i+1}}}{\varepsilon_{m_i}} = 2^{-\varepsilon(m_{i+1}-m_i)} \to 0$$

for $i \to \infty$, we deduce

$$\sup_m \frac{2^{-n}F(2^{-m})}{F(2^{-m-n})} \geq \frac{1}{4}2^{\varepsilon n},$$

which implies that $\beta_F \geq (1+\varepsilon)$. Hence $\beta_F = 1 + \varepsilon$ and this concludes the proof. ∎

Remark 1: Note that the above constructed universal spaces $\ell^{F_{\alpha,\beta}}(I)$ with indices α and β are neither β-concave nor α-convex. In general, it holds that a p-convex (resp. p-concave) space X with an uncountable symmetric basis containing an isomorphic copy of $\ell^p(\Gamma)$, for uncountable sets $\Gamma \subset I$, must be $X \equiv \ell^p(I)$. This follows from the impossibility of embedding isomorphically $\ell^1(\Gamma)$ into X for X different from $\ell^1(I)$ (Troyanski [T1]).

The α-convexity or the β-concavity conditions of the space $\ell^G(\Gamma)(I)$ in Theorem 11 cannot be removed. For instance, if $G(x) := F_{\alpha,\beta}(x)/|\log x|$ at 0, then the space $\ell^G(\Gamma)$ has indices $\alpha_G = \alpha$ and $\beta_G = \beta$ but it cannot be isomorphically embedded into the Orlicz space $\ell^{F_{\alpha,\beta}}(I)$ for uncountable sets $\Gamma \subset I$.

Remark 2: Notice that there is no uniqueness up to isomorphism of the universal space in the above theorem: We can construct, using the same method, an uncountable family of (non-isomorphic) Orlicz spaces $\ell^{F^{(a)}}(I)$ with indices α and β such that each space $\ell^{F^{(a)}}(I)$ is universal for the class of all α-convex β-concave Orlicz spaces $\ell^G(I)$. For instance, consider the family of universal functions $F^{(a)} = F^{(a)}_{\alpha,\beta}$, for $a \geq 4$, defined through Lemma 8 for the weight sequences $(c_k^a) = ((k+1)^a)$.

The above universal Orlicz function $F_{\alpha,\beta}$ with indices α and β at 0 is a *maximal* function inside the class of all Orlicz functions G at 0 with indices $\alpha < \alpha_G \leq \beta_G < \beta$ endowed with the order relation $H \leq G$ if and only if $H \in \sum_{G,1}$.

Remark 3: Theorem 11 provides also new examples of universal Orlicz *sequence* spaces $\ell^{F_{\alpha,\beta}} = \ell^F$ with prefixed indices α and β, which are different than Lindenstrauss and Tzafriri's examples ([L-T1], cf. [L-T2] Theorem 4.b.12). Indeed, for every α-convex β-concave Orlicz function G at 0 we have $G \in \Sigma_{F,1} \subset C_{F,1}$, so the Orlicz sequence space ℓ^G is isomorphic to a subspace H of ℓ^F (using [L-T2] Theorem 4.a.8). And, since this subspace H is generated by one vector in ℓ^F, it is uncomplemented in ℓ^F always so that the function G is not equivalent to F at 0 (cf. [L-T2] Theorem 3.a.9). This contrasts with the behavior of the universal sequence spaces ℓ^U given in [L-T1], where each space ℓ^G is isomorphic to a *complemented* subspace of ℓ^U (since each function $G \in E_U$).

The universal Orlicz sequence spaces $\ell^{U_{\alpha,\beta}} \equiv \ell^U$ constructed in ([L-T1]), in the particular case of choosing conjugate α and β, i.e. $1/\alpha + 1/\beta = 1$, provide examples of sequence spaces with a symmetric basis, different from ℓ^2, which are isomorphic to their own duals $(\ell^U)^*$. In the uncountable case, this cannot be done at all with the universal spaces $\ell^F(I)$ due to the uniqueness of the uncountable symmetric basis (Drewnowski [D]).

We pass now to construct scales of universal spaces $\ell^F(I)$ depending also on a third parameter γ, between α and β, that will give the associated lower inclusion index γ_F,

$$\gamma_F = \liminf_{x\to 0} \frac{\log F(x)}{\log x} = \sup\{p > 0 : \ell^p(I) \subset \ell^F(I)\}.$$

Recall that if $\ell^p(\Gamma)$ is isomorphically embedded into $\ell^F(I)$ for $\Gamma \subset I$ uncountable, then $\alpha_F \leq p \leq \gamma_F \leq \beta_F$. Notice that the universal spaces $\ell^{F_{\alpha,\beta}}(I)$ constructed in Theorem 11 satisfy $\gamma_{F_{\alpha,\beta}} = \beta$.

THEOREM 12: *Let $0 < \alpha < \gamma < \beta < \infty$. There exists an β-concave Orlicz space $\ell^{F_{\alpha,\beta,\gamma}}(I) \equiv \ell^F(I)$ with indices $\alpha_F = \alpha$, $\beta_F = \beta$ and $\gamma_F = \gamma$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^G(\Gamma)$ for sets $\Gamma \subset I$ and every α-convex β'-concave Orlicz function G at 0 with $\alpha \leq \beta' < \beta$ and $x^\gamma \prec G$ at 0.*

Proof: We can reduce to consider the case $\alpha = 1 < \gamma = 1 + \delta < \beta = 1 + \varepsilon$ for $0 < \delta < \varepsilon$. Let (ε_n) be the sequence considered in Lemma 10 and we define the functions f and F on $[0,1]$ by

$$f = \sum_{n=0}^{\infty} \varepsilon_n \chi_{(2^{-(n+1)},2^{-n}]} \quad \text{and} \quad F(x) = \int_0^x f(t)dt.$$

First let us assume that G is an strict convex Orlicz function at 0, so $G(x) = \int_0^x g(t)dt$ and $g(t)$ is a non-decreasing right continuous function with $g(0) = 0$ and $g(t) > 0$ for $t > 0$. Then, by Lemma 10, there exists a sequence (α'_n) of positive numbers and positive constants A and B such that for $1/2^{k+1} < x \leq 1/2^k$ we have

$$\begin{aligned} Ag(x) \leq Ag\Big(\frac{1}{2^k}\Big) &\leq \sum_{n=0}^{\infty} \alpha'_n \varepsilon_{n+k} = \sum_{n=0}^{\infty} \alpha'_n f\Big(\frac{x}{2^n}\Big) \\ &\leq B\Big(\sum_{i=0}^{k} \frac{g(2^{i-k})}{2^{i\varepsilon}} + \frac{1}{2^{k\delta}}\Big), \end{aligned}$$

where

$$\sum_{i=0}^{k} \frac{g(2^{i-k})}{2^{i\varepsilon}} \leq \sum_{i=0}^{\infty} \frac{g(2^{i-k})}{2^{i\varepsilon}} \leq \sum_{i=0}^{\infty} \frac{g(2^{i+1}x)}{2^{i\varepsilon}}.$$

Hence

$$Ag(x) \leq \sum_{n=0}^{\infty} \alpha'_n f\Big(\frac{x}{2^n}\Big) \leq B\Big(\sum_{i=0}^{\infty} \frac{g(2^{i+1}x)}{2^{i\varepsilon}} + 2^\delta x^\delta\Big)$$

for $0 \leq x \leq 1$. Then, by integration and the Beppo–Levi Theorem we have

$$AG(x) \leq \sum_{n=0}^{\infty} \alpha'_n 2^n F\Big(\frac{x}{2^n}\Big) \leq B\Big(\sum_{i=0}^{\infty} \frac{2^\varepsilon G(2^{i+1}x)}{2^{(i+1)(1+\varepsilon)}} + \frac{2^\delta x^{1+\delta}}{1+\delta}\Big)$$

for $0 < x \leq 1$. Now, by the $(1+\varepsilon')$-concavity of the function G at 0, for $0 < \varepsilon' < \varepsilon$, there exists a positive constant C such that $G(rx) \leq Cr^{(1+\varepsilon')}G(x)$ for $r > 1$ and $0 < x < 1$. Thus

$$\sum_{i=0}^{\infty} \frac{G(2^{i+1}x)}{2^{(i+1)(1+\varepsilon)}} \leq C'G(x)\Big(\sum_{i=0}^{\infty} \frac{1}{2^{(i+1)(\varepsilon-\varepsilon')}}\Big),$$

and using that $x^{\gamma} = x^{1+\delta} \leq CG(x)$, we deduce that there exist positive constants A_1 and B_1 such that for x near 0 we have

$$A_1 G(x) \leq \sum_{n=0}^{\infty} \alpha_n' 2^n F\Big(\frac{x}{2^n}\Big) \leq B_1 G(x).$$

And this implies, by ([R] Theorem B or [H-T] Proposition 5), that $\ell^F(I)$ contains an isomorphic copy of $\ell^G(\Gamma)$ for every set $\Gamma \subset I$.

The case of the function G a non-strict 1-convex function, i.e. when the function G is equivalent to x at 0 so $\ell^G(\Gamma) = \ell^1(\Gamma)$, can be obtained in a similar way as in Theorem 11 using now Lemma 3 of [H-R$_2$] instead of Lemma 8.

We can compute that the indices are $\alpha_F = 1$ and $\beta_F = 1 + \varepsilon$ by reasoning as in the proof of Theorem 11. Since $2^m F(2^{-m}) = \sum_{i=0}^{\infty}(\varepsilon_{m+i}/2^{i+1})$ and the sequence (ε_m) satisfies now $\varepsilon_m \leq 2^{n\varepsilon}\varepsilon_{m+n}$ for $m, n \in \mathbb{N}$, we have

$$\frac{2^{-n}F(2^{-m})}{F(2^{-m-n})} = \frac{\sum_{i=0}^{\infty} \frac{\varepsilon_{m+i}}{2^{i+1}}}{\sum_{i=0}^{\infty} \frac{\varepsilon_{m+n+i}}{2^{i+1}}} \leq 2^{n\varepsilon},$$

which implies that the function F is $(1+\varepsilon)$-concave at 0.

Finally, let us show that $\gamma_F = \gamma = 1 + \delta$. Since

$$2^{m_i}F(2^{-m_i}) > \varepsilon_{m_i}/2 = 2^{-m_i\delta}/2$$

we have that

$$\limsup_n \sqrt[n]{F(2^{-n})} \geq 2^{-(1+\delta)}.$$

On the other hand, we have

$$\begin{aligned} 2^n F(2^{-n}) = \sum_{k=0}^{\infty} \frac{\varepsilon_{n+k}}{2^{k+1}} &\leq \sum_{k=0}^{\infty} \frac{2^{-(n+k)\delta}}{2^{k+1}} \\ &\leq 2^{-n\delta}\frac{1}{(2-2^{-\delta})}, \end{aligned}$$

hence $\limsup_n \sqrt[n]{F(2^{-n})} \leq 2^{-(1+\delta)}$. Thus $\gamma_F = 1 + \delta = \gamma$, which concludes the proof. ∎

Remark: The condition $x^\gamma \prec G$ at 0 always holds when the Orlicz function G is γ-concave at 0, or when G satisfies

$$\delta_G = \limsup_{x\to 0} \frac{\log G(x)}{\log x} < \gamma.$$

Indeed, since $\gamma_G \le \delta_G < \gamma$, there exists p with $\delta_G < p < \gamma$. Then $x^\gamma \le x^p \le CG(x)$ for $0 \le x \le 1$ and some positive constant C.

Notice that the condition $x^\gamma \prec G$ at 0 does not imply in general any q-concavity of the Orlicz space $\ell^G(\Gamma)$, i.e. the Orlicz function G may fail the Δ_2-condition at 0 (cf. [K-R]).

4. Function spaces

We show now classes of lattice-universal separable Orlicz function spaces on $[0,1]$ with prefixed indices, using the previous results in the discrete case. Recall that for Orlicz function spaces the Boyd indices estimate also the grade of p-convexity and q-concavity of the space (cf. [L-T3] p. 139, [M]).

THEOREM 13: *Let $0 < \alpha < \beta < \infty$. There exists an Orlicz function space $L^{F_{\alpha,\beta}}[0,1] \equiv L^F[0,1]$, with indices $\alpha_F^\infty = \alpha$ and $\beta_F^\infty = \beta$, such that every α-convex β-concave Orlicz function space $L^G[0,1]$ is lattice-isomorphic to a sublattice of $L^F[0,1]$.*

Proof: Let $0 < \alpha < \beta < r < \infty$. We consider $\alpha_r = r - \beta$ and $\beta_r = r - \alpha$. It follows from Theorem 11 that there exists an Orlicz function F_r with indices at 0, $\alpha_{F_r} = \alpha_r$, and $\beta_{F_r} = \beta_r$ such that $\ell^{F_r}(I)$ is universal for the class of all α_r-convex and β_r-concave Orlicz function at 0. Now for every Orlicz functions G under the hypothesis of the theorem we define the function $G_r(x) := x^r G(1/x)$ for $0 < x < 1$. Then the function G_r is equivalent at 0 to an α_r-convex and β_r-concave function. Hence there exists a probability measure μ_{G_r} on $[0,1]$ such that the function $\widetilde{G}_r$, defined by

$$\widetilde{G}_r(x) = \int_0^1 \frac{F_r(xt)}{F_r(t)} d\mu_{G_r}(t)$$

for $0 \le x \le 1$, is equivalent to the function G_r at 0. Since $r > \beta_r$ we can assume w.l.o.g. that $F_r(xt) \ge x^r F(t)$ for $0 \le x, t \le 1$.

We consider now the non-decreasing function $F(x) = x^r F_r(1/x)$ for $x \ge 1$. We have the indices $\alpha_F^\infty = r - \beta_r = \alpha$ and $\beta_F^\infty = r - \alpha_r = \beta$, and the function $\widetilde{G}$, defined by $\widetilde{G}(x) = x^r \widetilde{G}_r(1/x)$, for $x \ge 1$, satisfies

$$\widetilde{G}(x) = \int_1^\infty \frac{F(xt)}{F(t)} d\mu(t) \quad (\text{for } x \ge 1)$$

where μ is a probability measure on $[1, +\infty)$ given by $\mu(t) = \mu_{G_r}(1/t)$, and $\widetilde{G}$ is equivalent to the function $x^r G_r(1/x) = G(x)$ at ∞. Thus $\int_0^\infty \frac{d\mu(t)}{F(t)} < \infty$ and $G \in \sum_{F,1}^{\infty}$, so by ([J-M-S-T] Theorem 7.7, cf. [H-R2]) we conclude that the space $L^F[0,1]$ contains a sublattice lattice-isomorphic to $L^G[0,1]$. ∎

Remark 1: Note that the above universal function space $L^{F_{\alpha,\beta}}[0,1]$, with indices α and β, is neither α-convex nor β-concave. In general, if a p-convex (resp. p-concave) rearrangement invariant function space X on $[0,1]$ with non-trivial concavity contains a sublattice lattice-isomorphic to $L^p[0,1]$, then $X[0,1] \equiv L^p[0,1]$. This follows from Theorem 3.2 of Kalton in ([K1]) and Theorem 7.7 in [H-K].

The α-convexity or β-concavity of the Orlicz function space $L^G[0,1]$ in the hypothesis of Theorem 13 cannot be removed. For instance, the function $G(x) := \frac{F_{\alpha,\beta}(x)}{\log(1+x)}$ at ∞ has indices $\alpha_G^\infty = \alpha$ and $\beta_G^\infty = \beta$ but the space $L^G[0,1]$ is not lattice-isomorphic to a sublattice of $L^{F_{\alpha,\beta}}[0,1]$.

Remark 2: In the case $1 < \alpha < \beta < 2$ the above Orlicz function space $L^{F_{\alpha,\beta}}[0,1] = L^F[0,1]$ with indices α and β is also universal for the class of all Orlicz function spaces $L^G(0,\infty)$ defined on the $(0,\infty)$-interval with Boyd indices strictly between α and 2. Indeed, $L^G(0,\infty)$ is isomorphic to a subspace of $L^r + L^2$ for $\alpha < r < 2$, which is isomorphic to $L^r[0,1]$ and hence embeds isomorphically into $L^F[0,1]$ (cf. [J-M-S-T] Theorem 8.6, [H-Ru]).

The universal spaces $L^{F_{\alpha,\beta}}[0,1]$ cannot be isomorphically embedded into the space $L^\alpha + L^\beta$ for $1 < \alpha < \beta < \infty$. This follows f.i. from Theorem 1.d.7 in [L-T2] in the case $1 < \alpha \le 2$ and from Theorem 4.2 in [H-K] for $\alpha > 2$.

Remark 3: If, via symmetrization, we follow the arguments used in ([L-T1]) in the construction of universal Orlicz sequence spaces in order to define Orlicz function spaces on $[0,1]$, we obtain function spaces $L^F[0,1]$ which are universal only for some classes of Orlicz *sequence* spaces (cf. Ruiz [Ru]).

Finally, we present scales of universal function spaces $L^F[0,1]$ depending also on a third parameter γ between α and β, which gives the upper inclusion index γ_F^∞,

$$\gamma_F^\infty = \limsup_{x\to\infty} \frac{\log F(x)}{\log x} = \inf\{p > 0 : L^p[0,1] \subset L^F[0,1]\}.$$

If $L^p[0,1]$ is lattice-isomorphic to a sublattice of $L^F[0,1]$ then $\gamma_F^\infty \le p \le \beta_F^\infty$. Notice that the universal spaces $L^{F_{\alpha,\beta}}[0,1]$ constructed in Theorem 13 satisfy $\gamma_{F_{\alpha,\beta}}^\infty = \alpha$.

THEOREM 14: *Let $0 < \alpha < \gamma < \beta < \infty$. There exists an α-convex Orlicz function space $L^{F_{\alpha,\beta,\gamma}}[0,1] \equiv L^F[0,1]$ with indices $\alpha_F^\infty = \alpha$, $\beta_F^\infty = \beta$ and $\gamma_F^\infty = \gamma$ such that $L^F[0,1]$ contains a sublattice lattice-isomorphic to $L^G[0,1]$ for every α'-convex β-concave Orlicz function G at ∞ with $\alpha < \alpha' \leq \beta$ and $x^\gamma \prec G$ at ∞.*

Proof: Let $0 < \alpha < \gamma < \beta < r < \infty$. We consider $\alpha_r = r - \beta$, $\beta_r' = r - \alpha'$, $\beta_r = r - \alpha$, and $\gamma_r = r - \gamma$, and using Theorem 12 we find a β_r-concave Orlicz function F_r with indices at 0, $\alpha_{F_r} = \alpha_r$, $\beta_{F_r} = \beta_r$ and $\gamma_{F_r} = \gamma_r$. Now given an Orlicz function G at ∞ under the hypothesis of the theorem, we consider the function defined at 0 by $G_r(x) = x^r G(1/x)$, which is α_r-convex and β_r'-concave at 0 and $x^{\gamma_r} \leq C G_r(x)$ for $0 \leq x \leq 1$. Then, by Theorem 12, there exists a probability measure μ_{G_r} on $[0,1]$ such that the function $\widetilde{G}_r(x)$ defined by

$$\widetilde{G}_r(x) = \int_0^1 \frac{F_0(xt)}{F_0(t)} d\mu_{G_r}(t) \quad (0 < x \leq 1)$$

is equivalent to the function G_r at 0.

Let us consider now the function $F(x) = x^r F_0(1/x)$ for $x \geq 1$. We have that $\alpha_F^\infty = r - \beta_r = \alpha$, $\gamma_F^\infty = r - \gamma_r = \gamma$ and $\beta_F^\infty = r - \alpha_r = \beta$ and that the function F is equivalent to an α-convex function at ∞. Now the function $\widetilde{G}(x) := x^r \widetilde{G}_r(1/x)$ for $x \geq 1$ satisfies

$$\widetilde{G}(x) = \int_1^\infty \frac{F(xt)}{F(t)} d\mu(t) \quad (\text{for } x \geq 1)$$

where μ is the probability measure on $[1, \infty)$ given by $\mu(t) = \mu_{G_r}(1/t)$, and $\widetilde{G}$ is equivalent at ∞ to the function $x^r G_r(1/x) = G(x)$. Thus, by ([J-M-S-T] Theorem 7.7), we conclude that $L^F[0,1]$ contains a sublattice lattice-isomorphic to $L^G[0,1]$. ∎

Remark: The condition $x^\gamma \prec G$ at ∞ holds when the Orlicz function G is γ-convex at ∞ or when G satisfies

$$\gamma < \delta_G^\infty = \liminf_{x \to \infty} \frac{\log G(x)}{\log x}.$$

Indeed, since $\gamma < \delta_G^\infty \leq \gamma_G^\infty$, there exists p with $\gamma < p < \delta_G^\infty$. Then $x^\gamma < x^p \leq CG(x)$ for $x \geq 1$ and some positive constant C.

Notice that the condition $x^\gamma \prec G$ at ∞ does not imply in general any q-convexity of the space $L^G[0,1]$ (cf. [L-T3] p. 140).

QUESTION: *We do not know whether for any Orlicz function space $L^F[0,1]$ the set of scalars p for which $L^p[0,1]$ is lattice-isomorphic to a sublattice of $L^F[0,1]$ is a convex set.*

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